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NONLINEAR STRESS–STRAIN EQUATIONS

RONALD L. HUSTON
University of Cincinnati, Cincinnati, Ohio

Abstract—Two unrelated methods which have previously been used in the development of stress–strain equations of nonlinear elasticity are examined. The first is the traditional notion of relating the stress to the strain through a strain energy function and the second is a recent geometrical approach suggested by Stojanovitch. These approaches are discussed and developed in a common notation making possible a comparison between them. This leads to explicit stress–strain relations in both the deformed and undeformed states of an elastic continuum. These relations, which are consistent with both approaches, are second order in the strain tensors and contain only two elastic constants.

THE DERIVATION of constitutive equations is clearly a fundamental issue in the theoretical development of nonlinear continuum mechanics. Truesdell [1] has provided a treatise which includes a discussion of the contributions of various writers toward the development of these equations. In nonlinear elasticity, these constitutive equations are the stress–strain equations, but here, explicit relationships have not been fully developed. The absence of such relationships has restricted the development of nonlinear elasticity since, in essence, only those problems in which the stress-strain equations can be left arbitrary, have been solved. Hence, the objective herein is to discuss and develop explicit relationships between the stress and the strain tensors.

Attempts to relate the stress and strain have been based primarily on thermodynamic considerations and on the introduction of a strain energy function which is usually left arbitrary. However, in 1960, Stojanovitch [2] proposed stress–strain equations based on ingenious geometrical considerations. In the discussions which follow these two approaches are reconciled and this provides a basis for the further development of the equations.

1. NOTATION AND PRELIMINARIES

The notation used is basically that of Truesdell and Toupin [3] and Eringen [4]. The elastic continuum is described in its undeformed or material state by the coordinates X^I and in its deformed or spatial state by the coordinates x^i. X^I and x^i are in general, curvilinear coordinates and there is a functional relationship between them depending upon the deformation. These coordinate systems are chosen independently in each state and thus they are not convected or dragged into each other as the convected coordinates used by Green and Zerna [5] and others.

In both the material and spatial states, contravariant and covariant base vectors may be introduced which in turn may be used to define contravariant and covariant metric tensors. If G_{IJ} and g_{ij} are the covariant material and spatial metric tensors respectively, the Cauchy and Green deformation tensors are given by the expressions,

$$c_{ij} = X^I_{,i} X^J_{,j} G_{IJ}, \qquad C_{IJ} = x^i_{,I} x^i_{,J} g_{ij}. \tag{1}$$

Using these deformation tensors, the spatial and material strain tensors, which are often called the Eulerian and Lagrangian strain tensors, are defined as,

$$2e_{ij} = g_{ij} - c_{ij}, \qquad 2E_{IJ} = C_{IJ} - G_{IJ}. \tag{2}$$

From equations (1) and (2) it is clear that these are related by the transformations,

$$E_{IJ} = x^i_{,I} x^i_{,J} e_{ij}, \qquad e_{ij} = X^I_{,i} X^J_{,j} E_{IJ}. \tag{3}$$

Although these strain tensors are covariant, metric tensors may be used to obtain the mixed and contravariant tensors,

$$\begin{aligned} E^I_J &= G^{IK} E_{KJ}, \qquad & E^{IJ} &= G^{IK} G^{JL} E_{KL} \\ e^i_j &= g^{ik} e_{kj}, \qquad & e^{ij} &= g^{ik} g^{jl} e_{kl}. \end{aligned} \tag{4}$$

These strain tensors, however, do not follow the simple transformation of equation (3), that is,

$$E^{IJ} \neq X^I_{,i} X^J_{,j} e^{ij}, \qquad e^{ij} \neq x^i_{,I} x^j_{,J} E^{IJ}. \tag{5}$$

Contravariant strains which do follow such a transformation may be obtained by introducing inverse deformation tensors, given by the expressions

$$(C^{-1})^{IJ} = X^I_{,i} X^J_{,j} g^{ij}, \qquad (c^{-1})^{ij} = x^i_{,I} x^j_{,J} G^{IJ}. \tag{6}$$

Then, analogous to equation (2), inverse Eulerian and Lagrangian strain tensors may be defined as,

$$2(e^{-1})^{ij} = g^{ij} - (c^{-1})^{ij}, \qquad 2(E^{-1})^{IJ} = (C^{-1})^{IJ} - G^{IJ}. \tag{7}$$

Thus, from equations (6) and (7), it is clear that these are related by the transformation,

$$(E^{-1})^{IJ} = X^I_{,i} X^J_{,j} (e^{-1})^{ij}, \qquad (e^{-1})^{ij} = x^i_{,I} x^j_{,J} (E^{-1})^{IJ}. \tag{8}$$

Mixed and covariant inverse strains may also be introduced as in equation (4). There are essentially, however, four basic strain tensors, those of equations (2) and (7). The former are clearly more conventional and are defined in a more natural manner. The latter have been introduced primarily for completeness and to help provide insight in the discussions which follow.

2. STRESS-STRAIN RELATIONS USING A STRAIN ENERGY FUNCTION

Parallel to the various strain tensors are several different stress tensors. Of these, perhaps the most conventional is the covariant spatial stress. Through thermodynamic and other considerations this stress may be related to the Eulerian strain by the equation

$$t^i_j = (\rho/\rho_0)(\delta^i_j - 2e^i_k)\, \partial\Sigma/\partial e^j_k \tag{9}$$

where t^i_j is a mixed spatial stress related to the covariant spatial stress by the relation,

$$t^i_j = g^{ik} t_{kj} \tag{10}$$

Σ is the strain energy function, ρ and ρ_0 are the mass densities of the spatial and material states, respectively, and δ^i_j is Kronecker's delta function. A derivation of equation (9) may be found in References [1, p. 179] and [4, p. 148]. The density ratio factor is unnecessary [3, p. 730], but it is conventionally included in the literature. This constitutive equation is merely one of several available, but it has been selected since the strain tensor occurs explicitly. Although it is based upon thermodynamic considerations, it is sometimes also regarded as a postulate or definition.

In its initial or undeformed state the elastic continuum is assumed to be stress free. Hence, material stress tensors, which are sometimes called pseudo stresses, are simply representations in material coordinates of the spatial stress. Of all the possible material stress tensors, one of the more conventional is Piola's contravariant stress T^{IJ} which is related to the contravariant spatial stress by the transformation [4, p. 109],

$$T^{IJ} = (\rho_0/\rho)X^I_{;i}X^J_{;j}t^{ij} \tag{11}$$

where

$$t^{ij} = g^{jl}t^i_l = g^{ik}g^{jl}t_{kl}. \tag{12}$$

Except for the density ratio factor, this transformation is identical to those relating the spatial and material strain tensors. Covariant and mixed material stress tensors may also be introduced by the relations,

$$T_{IJ} = G_{IK}T^K_J = G_{IK}G_{JL}T^{KL}. \tag{13}$$

These stresses, however, are not related to the analogous spatial stresses by simple transformations such as equation (11), that is,

$$T_{IJ} \neq (\rho_0/\rho)x^i_{;I}x^j_{;J}t_{ij}. \tag{14}$$

Let S_{IJ} represent a stress which does follow such a transformation, that is,

$$S_{IJ} = (\rho_0/\rho)x^i_{;I}x^j_{;J}t_{ij}. \tag{15}$$

By using equations (1), (11), (12), and (15), it is easy to show that S_{IJ} and T_{IJ} are related by the expression,

$$S_{IJ} = C_{IK}C_{JL}T^{KL}. \tag{16}$$

Analogous to equation (9), T^I_J is related to the Lagrangian strain by the equation [1, p. 177], [4, p. 146],

$$T^I_J = \partial\Sigma/\partial E^J_I. \tag{17}$$

In the references it is shown that this equation is equivalent to equation (9).

Assuming the elastic continuum to be homogeneous and isotropic, Σ is usually regarded as a function of the invariants of a strain tensor, and although the form of the function is usually not specified, some writers [1, p. 193] have suggested a power series expansion. In view of equations (9) and (17), the invariants of e^i_j and E^I_J are used. Of all the different forms these invariants may have, it is convenient to employ those used by Wesolowski [6] and also discussed by Ericksen [7]. These are given by the expressions,

$$\begin{aligned} \mathrm{I}_e &= e^k_k & \mathrm{I}_E &= E^K_K \\ \mathrm{II}_e &= e^k_l e^l_k & \mathrm{II}_E &= E^K_L E^L_K \\ \mathrm{III}_e &= e^k_l e^l_m e^m_k & \mathrm{III}_E &= E^K_L E^L_M E^M_K. \end{aligned} \tag{18}$$

As a series in these invariants, Σ may be represented as either

$$\begin{aligned}\Sigma = {} & a\mathrm{I}_e + b\mathrm{I}_e^2 + c\mathrm{II}_e + d\mathrm{I}_e^3 \\ & + e\mathrm{I}_e\mathrm{II}_e + f\mathrm{III}_e + \dots\end{aligned} \tag{19}$$

or

$$\begin{aligned}\Sigma = {} & A\mathrm{I}_E + B\mathrm{I}_E^2 + C\mathrm{II}_E + D\mathrm{I}_E^3 \\ & + E\mathrm{I}_E\mathrm{II}_E + F\mathrm{III}_E + \dots\end{aligned} \tag{20}$$

where the coefficients of the invariants are constants which depend upon the physical properties of the elastic continuum. Relations between these spatial and material coefficients may be obtained by expressing the material invariants in terms of the spatial invariants. This is possible since e^i_j and E^I_J are related through equations (3) and (4). Algebraic manipulation of these and the other preliminary equations of the preceding section lead to the expressions,

$$\begin{aligned}\mathrm{I}_E &= \mathrm{I}_e + 2\mathrm{II}_e + 4\mathrm{III}_e + \dots \\ \mathrm{II}_E &= \mathrm{II}_e + 4\mathrm{III}_e + \dots \\ \mathrm{III}_E &= \mathrm{III}_e + \dots\end{aligned} \tag{21}$$

where, as in equations (19) and (20), the terms not written are of the order of the fourth power of the strain. Hence, substituting equations (21) into (20) and comparing with (19), the following relations between the coefficients are obtained,

$$\begin{aligned} a &= A & d &= D \\ b &= B & e &= E + 4B \\ c &= C + 2A & f &= F + 4C + 4A.\end{aligned} \tag{22}$$

The series in equations (19) and (20) are clearly ascending in orders of the strain. If the series are terminated as shown, equations (9) and (17) provide stress–strain relations which are second order approximations. Hence, upon substitution of equations (19), and (20), (9) and (7) become,

$$\begin{aligned}t^i_j = {} & (\rho/\rho_0)[(a + 2b\mathrm{I}_e + 3d\mathrm{I}_e^2 + e\mathrm{II}_e)\delta^i_j \\ & + (2c - 2a + 2e\mathrm{I}_e - 4b\mathrm{I}_e)e^i_j \\ & + (3f - 4c)e^i_k e^k_j]\end{aligned} \tag{23}$$

and

$$\begin{aligned}T^I_J = {} & (A + 2B\mathrm{I}_E + 3D\mathrm{I}_E^2 + E\mathrm{II}_E)\delta^I_J \\ & + (2C + 2E\mathrm{I}_E)E^I_J \\ & + 3FE^I_K E^K_J\end{aligned} \tag{24}$$

where terms of the third order in the strain have been dropped. Using equations (2),

(16) and (24), the covariant material stress S_{IJ} can thus be expressed as,

$$S_{IJ} = (A+2B\mathrm{I}_E+E\mathrm{II}_E+3D\mathrm{I}_E^2)G_{IJ} +(4A+2C+2E\mathrm{I}_E+8B\mathrm{I}_E)E_{IJ} +(2A+8C+3F)E_{IK}E_J^K. \tag{25}$$

In equation (2), the density ratio factor ρ/ρ_0 may be approximated to the second order in the strain, by the expression [3, p. 267], [7],

$$\rho/\rho_0 = 1-\mathrm{I}_e+\tfrac{1}{2}\mathrm{I}_e^2-\mathrm{II}_e. \tag{26}$$

Hence, equation (20) becomes,

$$t_j^i = [a+(2b-a)\mathrm{I}_e+(3d-2b)\mathrm{I}_e^2 +(2a+e)\mathrm{II}_e]\delta_j^i+[2c-2a +(2a+2e-4b-2c)\mathrm{I}_e]e_j^i +[3f-4c]e_k^i e_j^k. \tag{27}$$

Equations (23) and (24) contain six undetermined elastic constants and several writers [1, pp. 201–211] have attempted, through various means, to relate them (that is, analogous constants) to the familiar Lamé constants λ and μ. In the following section stress–strain equations are developed in a completely different manner and then by comparison, these constants are identified.

3. STRESS–STRAIN RELATIONS BASED ON GEOMETRICAL CONSIDERATIONS

Since the elastic continuum is homogeneous and isotropic, it can be represented in the material and spatial states by the fourth order isotropic tensors,

$$L_{IJKL} = \alpha G_{IJ}G_{KL}+\beta(G_{IK}G_{JL}+G_{IL}G_{JK}) \tag{28}$$

and

$$l_{ijkl} = \alpha g_{ij}g_{kl}+\beta(g_{ik}g_{jl}+g_{il}g_{jk}) \tag{29}$$

where α and β are constants depending upon the physical properties of the elastic continuum. Equations (28) and (29) are obtained by transforming into curvilinear coordinates the expressions found in References [8] and [9]. Analogous to the definitions of the deformation tensors in equation (1), fourth order deformation tensors may be defined as,

$$m_{ijkl} = X_{,i}^I X_{,j}^J X_{,k}^K X_{,l}^L L_{IJKL} \tag{30}$$

and

$$M_{IJKL} = x_{,I}^i x_{,J}^j x_{,K}^k x_{,L}^l l_{ijkl}. \tag{31}$$

Substituting equations (28) and (29) into (30) and (31), respectively, and using equation (1) then leads to the relations,

$$m_{ijkl} = \alpha c_{ij}c_{kl}+\beta(c_{ik}c_{jl}+c_{il}c_{jk}) \tag{32}$$

and

$$M_{IJKL} = \alpha C_{IJ}C_{KL} + \beta(C_{IK}C_{JL} + C_{IL}C_{JK}). \tag{33}$$

Following the suggestion of Stojanovitch [2] and the analogy of the definition of the Lagrangian and Eulerian strain in equation (2), fourth order material and spatial stress tensors are defined as,

$$T_{IJKL} = M_{IJKL} - L_{IJKL} \tag{34}$$

and

$$t_{ijkl} = (\rho/\rho_0)(l_{ijkl} - m_{ijkl}) \tag{35}$$

where, as in equation (9), the density ratio factor is conventionally included. Noting equations (30) and (31), it is clear that these stress tensors are related by the transformation,

$$T_{IJKL} = (\rho_0/\rho)x^i_{;I}x^j_{;J}x^k_{;K}x^l_{;L}t_{ijkl}. \tag{36}$$

The second order covariant spatial stress is defined as the contracted tensor,

$$t_{ij} = g^{kl}t_{ijkl}. \tag{37}$$

Using equations (2), (29), (32), and (35), this becomes,

$$\begin{aligned} t_{ij} = (\rho/\rho_0)[&2\alpha \mathrm{I}_e g_{ij} + (6\alpha + 8\beta)e_{ij} \\ &- 4\alpha \mathrm{I}_e e_{ij} - 8\beta e_{ik}e^k_j]. \end{aligned} \tag{38}$$

If this stress tensor is identified with the spatial stress tensor of the previous section, the material stress S_{IJ}, upon using equations (15), (36), and (37), may be expressed as,

$$S_{IJ} = (C^{-1})^{KM}T_{IJKM}. \tag{39}$$

Using equations (7), (28), and (34), this becomes, after some algebraic manipulations, to the second order in the strain,

$$\begin{aligned} S_{IJ} = &2\alpha \mathrm{I}_E G_{IJ} + (6\alpha + 8\beta)E_{IJ} \\ &- 4\alpha \mathrm{II}_E G_{IJ} - 8\beta E_{IK}E^K_J. \end{aligned} \tag{40}$$

Comparing equations (23) and (38) leads to the following relations between the physical constants,

$$\begin{aligned} a &= 0 & e &= 0 & 3f - 4c &= -8\beta \\ 2b &= 2\alpha & 2c - 2a &= 6\alpha + 8\beta & & \\ 3d &= 0 & 2e - 4b &= -4\alpha. & & \end{aligned} \tag{41}$$

In like manner, comparing equations (25) and (40) leads to the expressions,

$$\begin{aligned} A &= 0 & 3D &= 0 & 2A + 8C + 3F &= -8\beta \\ 2B &= 2\alpha & 4A + 2C &= 6\alpha + 8\beta & & \\ E &= -4\alpha & 2E + 8B &= 0. & & \end{aligned} \tag{42}$$

In linear elasticity there is no distinction between the spatial and material coordinates and the stress–strain equations take the familiar form,

$$t^i_j = \lambda \mathrm{I}_e \delta^i_j + 2\mu e^i_j. \tag{43}$$

By neglecting all but the terms linear in the strain, equation (38) becomes,

$$t_{ij} = 2\alpha \mathrm{I}_e g_{ij} + (6\alpha + 8\beta) e_{ij}. \tag{44}$$

Comparing these equations identifies α and β as,

$$\alpha = \lambda/2, \qquad \beta = (2\mu - 3\lambda)/8. \tag{45}$$

Hence, from equations (39) and (40), the spatial and material constants are given by,

$$\begin{aligned} a = d = e = 0, &\qquad c = \mu \\ b = \lambda/2, &\qquad f = \lambda + (2/3)\mu \end{aligned} \tag{46}$$

and

$$\begin{aligned} A = D = 0, &\qquad C = \mu \\ B = \lambda/2 &\qquad E = -2\lambda \\ &\qquad F = \lambda - (10/3)\mu. \end{aligned} \tag{47}$$

Using these results, the strain energy Σ in equations (19) and (20) may be expressed to the third order in the strain as,

$$\sum = (\lambda/2)\mathrm{I}_e^2 + \mu \mathrm{II}_e + [\lambda + (2/3)\mu]\mathrm{III}_e \tag{48}$$

and

$$\sum = (\lambda/2)\mathrm{I}_E^2 + \mu \mathrm{II}_E - 2\lambda \mathrm{I}_E \mathrm{II}_E + [\lambda - (10/3)\mu]\mathrm{III}_E. \tag{49}$$

Finally, the spatial and material stress strain equations (23) and (25) or (38) and (40) take the form,

$$\begin{aligned} t_{ij} = (\rho/\rho_0)[&\lambda \mathrm{I}_e g_{ij} + 2\mu e_{ij} \\ &- 2\lambda \mathrm{I}_e e_{ij} + (3\lambda - 2\mu) e_{ik} e^k_j] \end{aligned} \tag{50}$$

and

$$\begin{aligned} S_{IJ} = &\lambda \mathrm{I}_E G_{IJ} + 2\mu E_{IJ} \\ &- 2\lambda \mathrm{II}_E G_{IJ} + (3\lambda - 2\mu) E_{IK} E^K_J. \end{aligned} \tag{51}$$

4. CONCLUSIONS

The comparison of the two diverse approaches toward the development of the stress–strain equations has in equations (41) and (42) led to the identification of the spatial and material physical constants of the spatial and material versions of the strain energy function. It is interesting to note that whereas these equations each provide seven relations for six unknowns, they are not inconsistent. Also, these spatial and material constants identified in equations (46) and (47) identically satisfy the relations between them established in equation (22). Finally, the resulting equations (50) and (51), consistent with both approaches, provide explicit second order stress–strain relationships, which can be significantly useful in nonlinear elasticity.

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Résumé—Deux méthodes n'ayant aucune relation entre elles, et qui ont été employées auparavant pour le développement des équations effort–tension d'élasticité non-linéaire, y sont étudiées. La première représente la notion classique qui relie l'effort à la tension par une fonction d'énergie de tension, et la seconde consiste à aborder la question géométriquement, selon Stojanovitch. Ces abords sont discutés et développés au moyen d'une même notation, ce qui permet de les comparer. On en est conduit à des relations explicites entre effort et tension, tant soit à l'état déformé que non-déformé, d'un continuum élastique. Ces relations, vérifiables par les deux méthodes, sont de second ordre dans les tenseurs de tension et ne contiennent que deux constantes d'élasticité.

Zusammenfassung—Zwei voneinander unabhängige Methoden, welche früher für die Entwicklung der Spannungs-Dehnungs-Gleichungen nicht-linearer Elastizität angewandt wurden, werden überprüft. Die erste beruht auf der traditionellen Annahme der Abhängigkeit der Spannung von Dehnung unter Zuhilfenahme einer Dehnungsenergie Funktion, und die zweite ist eine neuere geometrische Annäherung welche von Stojanovitch vorgeschlagen wurde.

Diese Annäherungen wurden an Hand gemeinsamer Aufzeichnungen erörtert und entwickelt um einen Vergleich zwischen ihnen zu ermöglichen. Dies führt zu eingehenden Spannungs-Dehnungsbeziehungen für deformierte und undeformierte Arten des Zustandes eines elastischen Kontinuums. Diese Beziehungen welche mit beiden Annäherungen vereinbart werden können sind zweiter Ordnung in Dehntensoren und beinhalten bloss zwei elastische Konstante.

Абстракт—Рассмотренны два независимых метода раньше применявшихся дла развития уравнений "напряжение—деформация" нелинейной упругости. Первый их них—это традиционная идея сопоставления напряжения и деформации посредством функции энергии деформации, а второй—это геометрический подход предложенный Стояновичем. Оба метода рассмотренны и разработанны при помощи общей системы обозначений, что позволяет взаимное сравнение. Это приводит к выявлению соотношения напряжения и деформации как в деформированном так и в недеформированном состоянии эластического континуума. Эти соотношения (совместимые с обоими методами) являются соотношениями второго порядка тензора деформации и включают всего лишь две постоянных эластичности.